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Higher Mathematics

UNIT 2 OUTCOME 4

Circles

Contents

HSN22400

OUTCOME 4

Circles

1 Representing a Circle

The equation of a circle with centre (a, b) and radius r units is

$$(x-a)^2+(y-b)^2=r^2.$$

This is given in the exam.

For example, the circle with centre $(2, -1)$ and radius 4 units has equation:

$$(x-2)^2+(y+1)^2=4^2$$
$$(x-2)^2+(y+1)^2=16.$$

Note that the equation of a circle with centre $(0,0)$ is of the form $x^2+y^2=r^2$, where r is the radius of the circle.

EXAMPLES

1. Find the equation of the circle with centre $(1, -3)$ and radius $\sqrt{3}$ units.

2. A is the point $(-3,1)$ and $B(5,3)$.

   Find the equation of the circle which has AB as a diameter.

2 Testing a Point

Given a circle with centre (a, b) and radius r units, we can determine whether a point (p, q) lies within, outwith or on the circumference using the following rules:

$(p-a)^2 + (q-b)^2 < r^2 \Leftrightarrow$ the point lies within the circle

$(p-a)^2 + (q-b)^2 = r^2 \Leftrightarrow$ the point lies on the circumference of the circle

$(p-a)^2 + (q-b)^2 > r^2 \Leftrightarrow$ the point lies outwith the circle.

EXAMPLE

A circle has the equation $(x-2)^2 + (y+5)^2 = 29$.

Determine whether the points $(2, 1)$, $(7, -3)$ and $(3, -4)$ lie within, outwith or on the circumference of the circle.

3 The General Equation of a Circle

The equation of any circle can be written in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where the centre is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$ units.

This is given in the exam.

Note that the above equation only represents a circle if $g^2 + f^2 - c > 0$, since:

- if $g^2 + f^2 - c < 0$ then we cannot obtain a real value for the radius, since we would have to square root a negative;
- if $g^2 + f^2 - c = 0$ then the radius is zero – the equation represents a point rather than a circle.

EXAMPLES

1. Find the radius and centre of the circle with equation $x^2 + y^2 + 4x - 8y + 7 = 0$.

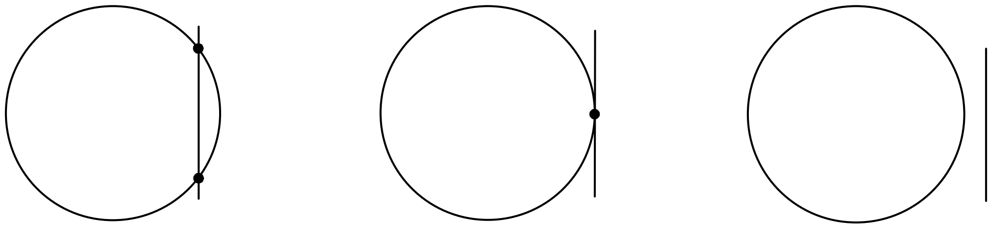
2. Find the radius and centre of the circle with equation $2x^2 + 2y^2 - 6x + 10y - 2 = 0$.

3. Explain why $x^2 + y^2 + 4x - 8y + 29 = 0$ is not the equation of a circle.

4. For which values of k does $x^2 + y^2 - 2kx - 4y + k^2 + k - 4 = 0$ represent a circle?

4 Intersection of a Line and a Circle

A straight line and circle can have two, one or no points of intersection:



two intersections one intersection no intersections

If a line and a circle only touch at one point, then the line is a **tangent** to the circle at that point.

To find out how many times a line and circle meet, we can use substitution.

EXAMPLES

1. Find the points where the line with equation $y = 3x$ intersects the circle with equation $x^2 + y^2 = 20$.

2. Find the points where the line with equation $y = 2x + 6$ and circle with equation $x^2 + y^2 + 2x + 2y - 8 = 0$ intersect.

5 Tangents to Circles

As we have seen, a line is a tangent if it intersects the circle at only one point.

To show that a line is a tangent to a circle, the equation of the line can be substituted into the equation of the circle, and solved – there should only be one solution.

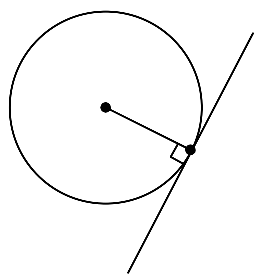
EXAMPLE

Show that the line with equation $x + y = 4$ is a tangent to the circle with equation $x^2 + y^2 + 6x + 2y - 22 = 0$.

6 Equations of Tangents to Circles

If the point of contact between a circle and a tangent is known, then the equation of the tangent can be calculated.

If a line is a tangent to a circle, then a radius will meet the tangent at right angles. The gradient of this radius can be calculated, since the centre and point of contact are known.



Using $m_{\text{radius}} \times m_{\text{tangent}} = -1$, the gradient of the tangent can be found.

The equation can then be found using $y - b = m(x - a)$, since the point is known, and the gradient has just been calculated.

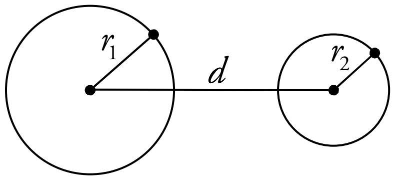
EXAMPLE

Show that $A(1, 3)$ lies on the circle $x^2 + y^2 + 6x + 2y - 22 = 0$ and find the equation of the tangent at A.

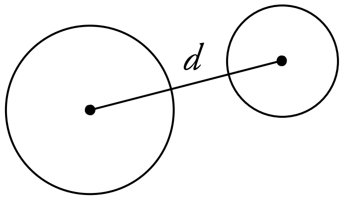
7 Intersection of Circles

Consider two circles with radii r_1 and r_2 with $r_1 > r_2$.

Let d be the distance between the centres of the two circles.

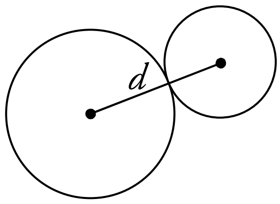


$d > r_1 + r_2$



The circles do not touch.

$d = r_1 + r_2$

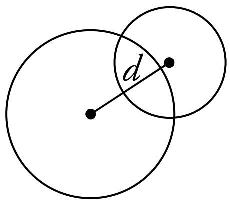


The circles touch externally.

Note

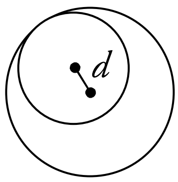
Don't try to memorise this, just try to understand why each one is true.

$r_1 - r_2 < d < r_1 + r_2$



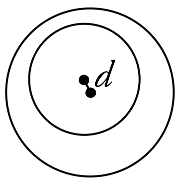
The circles meet at two distinct points.

$d = r_1 - r_2$



The circles touch internally.

$d < r_1 - r_2$



The circles do not touch.

EXAMPLES

1. Circle P has centre $(-4, -1)$ and radius 2 units, circle Q has equation $x^2 + y^2 - 2x + 6y + 1 = 0$. Show that the circles P and Q do not touch.

2. Circle R has equation $x^2 + y^2 - 2x - 4y - 4 = 0$, and circle S has equation $(x-4)^2 + (y-6)^2 = 4$. Show that the circles R and S touch externally.