Higher

Mathematics

UNIT 3 OUTCOME 2

Further Calculus

Contents

HSN23200

OUTCOME 2

Further Calculus

1 Differentiating sin*x* and cos*x*

In order to differentiate expressions involving trigonometric functions, we use the following rules:

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

These rules only work when x is an angle measured in radians. A form of these rules is given in the exam.

EXAMPLES

1. Differentiate $y = 3\sin x$ with respect to x.

2. A function f is defined by $f(x) = \sin x - 2\cos x$ for $x \in \mathbb{R}$.

   Find $f'\left(\frac{\pi}{3}\right)$.

3. Find the equation of the tangent to the curve $y = \sin x$ when $x = \frac{\pi}{6}$.

2 Integrating sinx and cosx

We know the derivatives of $\sin x$ and $\cos x$, so it follows that the integrals are:

$$\int \cos x \; dx = \sin x + c, \qquad \int \sin x \; dx = -\cos x + c.$$

Again, these results only hold if x is measured in radians.

EXAMPLES

1. Find $\int (5\sin x + 2\cos x) \; dx$.

2. Find $\int_0^{\frac{\pi}{4}} (4\cos x + 2\sin x)\, dx$.

3. Find the value of $\int_0^4 \frac{1}{2}\sin x \, dx$.

Remember
We must use radians when integrating or differentiating trig. functions.

3 The Chain Rule

We will now look at how to differentiate composite functions, such as $f(g(x))$. If the functions f and g are defined on suitable domains, then

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \times g'(x).$$

Stated simply: differentiate the outer function, the bracket stays the same, then multiply by the derivative of the bracket.

This is called the **chain rule**. You will need to remember it for the exam.

EXAMPLE

If $y = \cos\left(5x + \frac{\pi}{6}\right)$, find $\frac{dy}{dx}$.

4 Special Cases of the Chain Rule

We will now look at how the chain rule can be applied to particular types of expression.

Powers of a Function

For expressions of the form $\left[f(x)\right]^n$, where n is a constant, we can use a simpler version of the chain rule:

$$\frac{d}{dx}\left[\left(f(x)\right)^n\right] = n\left[f(x)\right]^{n-1} \times f'(x).$$

Stated simply: the power (n) multiplies to the front, the bracket stays the same, the power lowers by one (giving $n-1$) and everything is multiplied by the derivative of the bracket $(f'(x))$.

EXAMPLES

1. A function f is defined on a suitable domain by $f(x)=\sqrt{2x^2+3x}$.
   Find $f'(x)$.

2. Differentiate $y=2\sin^4 x$ with respect to x.

Powers of a Linear Function

The rule for differentiating an expression of the form $(ax+b)^n$, where a, b and n are constants, is as follows:

$$\frac{d}{dx}\left[(ax+b)^n\right]=an(ax+b)^{n-1}.$$

EXAMPLES

3. Differentiate $y=(5x+2)^3$ with respect to x.

4. If $y = \dfrac{1}{(2x+6)^3}$, find $\dfrac{dy}{dx}$.

5. A function f is defined by $f(x) = \sqrt[3]{(3x-2)^4}$ for $x \in \mathbb{R}$. Find $f'(x)$.

Trigonometric Functions

The following rules can be used to differentiate trigonometric functions.

$$\frac{d}{dx}\left[\sin(ax+b)\right] = a\cos(ax+b) \qquad \frac{d}{dx}\left[\cos(ax+b)\right] = -a\sin(ax+b)$$

These are given in the exam.

EXAMPLE

6. Differentiate $y = \sin(9x+\pi)$ with respect to x.

5 A Special Integral

The method for integrating an expression of the form $(ax+b)^n$ is:

$$\int (ax+b)^n \ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \qquad \text{where } a \neq 0 \text{ and } n \neq -1.$$

Stated simply: raise the power (n) by one, divide by the new power and also divide by the derivative of the bracket ($a(n+1)$), add c.

EXAMPLES

1. Find $\int (x+4)^7 \ dx$.

2. Find $\int (2x+3)^2 \ dx$.

3. Find $\int \frac{1}{\sqrt[3]{5x+9}} \ dx$ where $x \neq -\frac{9}{5}$.

4. Evaluate $\int_0^3 \sqrt{3x+4}\, dx$ where $x \geq -\frac{4}{3}$.

Warning

Make sure you don't confuse differentiation and integration – this could lose you a lot of marks in the exam.

Remember the following rules for differentiating and integrating expressions of the form $(ax+b)^n$:

$$\frac{d}{dx}\left[(ax+b)^n\right] = an(ax+b)^{n-1},$$

$$\int (ax+b)^n \; dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$$

These rules will *not* be given in the exam.

Using Differentiation to Integrate

Recall that integration is just the process of undoing differentiation. So if we differentiate $f(x)$ to get $g(x)$ then we know that $\int g(x)\,dx = f(x)+c$.

EXAMPLES

5. (a) Differentiate $y=\dfrac{5}{(3x-1)^4}$ with respect to x.

   (b) Hence, or otherwise, find $\displaystyle\int \frac{1}{(3x-1)^5}\,dx$.

6. (a) Differentiate $y=\dfrac{1}{\left(x^3-1\right)^5}$ with respect to x.

   (b) Hence, find $\displaystyle\int \frac{x^2}{\left(x^3-1\right)^6}\,dx$.

6 Integrating sin(*ax* + *b*) and cos(*ax* + *b*)

Since we know the derivatives of $\sin(ax+b)$ and $\cos(ax+b)$, it follows that their integrals are:

$$\int \cos(ax+b)\ dx = \tfrac{1}{a}\sin(ax+b)+c,$$

$$\int \sin(ax+b)\ dx = -\tfrac{1}{a}\cos(ax+b)+c.$$

These are given in the exam.

EXAMPLES

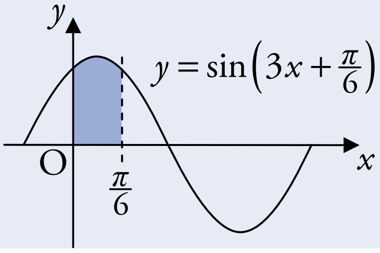
1. Find $\int \sin(4x+1)\ dx$.

2. Find $\int \cos\left(\tfrac{3}{2}x+\tfrac{\pi}{5}\right)\ dx$.

3. Find the value of $\int_0^1 \cos(2x-5)\ dx$.

Remember
We must use radians when integrating or differentiating trig. functions.

4. Find the area enclosed by the graph of $y = \sin\left(3x + \frac{\pi}{6}\right)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$.



5. Find $\int 2\cos\left(\frac{1}{2}x - 3\right)\, dx$.

6. Find $\int \left(5\cos(2x) + \sin\left(x - \sqrt{3}\right)\right)\, dx$.

7. (a) Differentiate $\frac{1}{\cos x}$ with respect to x.

(b) Hence find $\int \frac{\tan x}{\cos x}\, dx$.